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ON CIRCUIT INTEGRATION OVER A STRAIGHT LINE.

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1. In the Annals of Mathematics, Vol. VIII, p. 137, I attempted to utilize Cauchy's method of integrating a function of a complex variable around a closed boundary containing a pole of the function, with the purpose in view of devising a similar method for a function of a real variable by integrating forward and backward to the place of beginning along the axis of the variable. I have since then seen the error in the fundamental formula in the paper referred to, and wish now to make that acknowledgment and to notice some interesting points in connection with the investigations connected with it.

2. We observe that $(t - a)^{-1}$ is the derivative of $\log (a - t)$ when $t < a$, and the derivative of $\log (t - a)$ when $t > a$. Hence

$$\int_{a-\alpha}^{a-\beta} \frac{dt}{t-a} = \log \frac{\beta}{\alpha}; \quad \int_{a+\beta}^{a+\alpha} \frac{dt}{t-a} = \log \frac{\alpha}{\beta}.$$

Therefore

$$\int_{a+\beta}^{a+\alpha} \frac{dt}{t-a} - \int_{a-\alpha}^{a-\beta} \frac{dt}{t-a} = \int_{a+\beta}^{a+\alpha} \frac{dt}{t-a} + \int_{a-\beta}^{a-\alpha} \frac{dt}{t-a},$$

$$= \log \left[\frac{\alpha}{\beta}\right]^2.$$

Let $q > p$ be two numbers defining an interval $((p, q))$ containing a number a. Let α and β be two absolute numbers such that $\beta < \alpha$ and $a \pm \alpha$, $a \pm \beta$ are also in $((p, q))$. We call the indefinite *circuit* integral of a function over $((p, q))$, the sum of the four ordinary integrals taken from p to $a - \alpha$, from $a + \beta$ to q, from q to $a + \alpha$, from $a - \beta$ to p. Obviously, the intervals $((p, a - \alpha))$, $((a + \alpha, q))$ are passed over twice, once forward and once backward. The intervals $((a - \alpha, a - \beta))$, $((a + \beta, a + \alpha))$ are passed over once backward and forward respectively. While the gap $(a - \beta, a + \beta)$ is not entered upon at all by the variable.

The indefinite circuit integral of any function y that is uniform and continuous everywhere in $((p, q))$ except at the point a, is evidently

$$\int_{(pq)} ydx = \int_{a-\beta}^{a-\alpha} ydx + \int_{a+\beta}^{a+\alpha} ydx.$$

The indefinite circuit integral of $(t - a)^{-1}$ over $((p, q))$ is, by the above, equal to $\log (\alpha/\beta)^2$. This is indefinite until the quotient α/β is assigned. We agree now once for all that the quotient $(\alpha/\beta)^2$ shall be e for all values of α and β, and we let α and β converge to zero. When this is the case, the limit to which converges the indefinite circuit integral as α and β converge to zero is called the *definite* circuit integral with respect to a, or briefly the circuit integral with respect to a over $((p, q))$. Thus

$$\int_{(pq)} \frac{dt}{t - a} = 1 .$$

3. Let $f(x)$ be any one-valued and continuous function at *all* points in $((p, q))$. Clearly, the circuit integral of $f(x)$ over $((p, q))$ is zero. But,

$$\begin{aligned} \int_{(pq)} \frac{f(t)}{t - a}\, dt &= \int_{(pq)} \frac{f(t) - f(a)}{t - a}\, dt + f(a) \int_{(pq)} \frac{dt}{t - a}, \\ &= f(a), \end{aligned} \tag{1}$$

if $f'(a)$ is an assigned number. Otherwise, since $t - a$ is always positive in $((a + \beta, a + \alpha))$ and negative in $((a - \alpha, a - \beta))$, we have for the circuit integral

$$\int_{(pq)} \frac{f(t)dt}{t - a} = \mathcal{L}_{\alpha=0} \left\{ f(t') \int_{a-\beta}^{a-\alpha} \frac{dt}{t - a} + f(t'') \int_{a+\beta}^{a+\alpha} \frac{dt}{t - a} \right\},$$

where t' and t'' are numbers in $((a - \alpha, a - \beta))$, $((a + \beta, a + \alpha))$ respectively, and which converge to a for $\alpha = 0$, while the limit of each of the integrals is $\log e^{\frac{1}{2}}$. Therefore the circuit integral is $f(a)$.

Let a_1 and a_2 be two points in $((p, q))$. The circuit integral with respect to a_1 and a_2 over $((p, q))$ is the sum of the circuit interval with respect to a_1 over $((p, r))$ and the circuit integral with respect to a_2 over $((r, q))$, where r is any number between a_1 and a_2. The generalization with respect to n points $a_1, \ldots, a_n$ in $((p, q))$ is obvious. We have

$$\begin{aligned} \int_{(pq)} \frac{f(t)dt}{(t - a_1)(t - a_2)} &= \int_{(pr)} \frac{f(t)}{t - a_2} \frac{dt}{t - a_1} + \int_{(rq)} \frac{f(t)}{t - a_1} \frac{dt}{t - a_2}, \\ &= \frac{f(a_1)}{a_2 - a_1} + \frac{f(a_2)}{a_1 - a_2} = \frac{f(a_1) - f(a_2)}{a_1 - a_2}. \end{aligned} \tag{2}$$

In general, it follows at once that

$$\int_{(pq)} \frac{f(t)dt}{(t - a_1) \ldots (t - a_n)} = \sum_{r=1}^{n} \frac{f(a_r)}{\psi'(a_r)},$$

where $\psi'(t)$ is the derivative of $(t-a_1)\dots(t-a_n)$. If $f(x)=\psi'(x)$, this integral is n.

4. Again, by the above, we have

$$\int_{(pq)} \frac{f(t)dt}{(t-x)(t-a_1)\dots(t-a_n)} = \frac{f(x)}{\psi(x)} - \sum_{1}^{n} \frac{f(a_r)}{\psi'(a_r)}\frac{1}{x-a_r}. \tag{3}$$

Therefore

$$\psi(x)\int_{(pq)} \frac{f(t)dt}{(t-x)\,\psi(t)} = f(x) - \sum_{1}^{n} \frac{f(a_r)}{\psi'(a_r)}\frac{\psi(x)}{x-a_r}. \tag{4}$$

The function on the right vanishes when $x=a_1,\dots,a_n$, and the term under the sigma is a rational integral function of x of degree $n-1$. It is therefore equal to

$$\frac{(x-a_1)\dots(x-a_n)}{n!}f^n(u),$$

where u is some number between the greatest and least of the numbers x, a_1, $\dots$, a_n, and we have

$$\int_{(pq)} \frac{f(t)dt}{(t-x)\,\psi(t)} = \frac{f^n(u)}{r!}, \tag{5}$$

provided $f(x)$ admits of differentiation and not unless.

If $f(x)$ is a rational integral function of degree less than n, the integral is zero.

5. Again, let

$$\Delta^n f(x) = \sum_{r=n}^{0}(-1)^{n-r}C_{n,r}f(x+rh),$$

$$(x-a)^{(n+1)} = (x-a)(x-a-h)\dots(x-a-nh),$$

$x-a-rh$, $(r=0,\dots,n)$, being points in (p,q). Then

$$\int_{(pq)} \frac{f(t)dt}{(t-a)^{(n+1)}} = \sum_{n}^{0} \frac{f(a+rh)}{(rh)(r-1h)\dots h(-h)\dots(-r-nh)},$$

$$= \sum_{n}^{0}(-1)^{n-r}\frac{n!}{r!\,(n-r)!}\frac{f(a+rh)}{h^n},$$

$$= \frac{1}{n!}\frac{\Delta^n f(a)}{h^n} = \frac{f^n(u)}{n!}, \tag{6}$$

when u is between a and $a+nh$.

6. Consider the integral (4) when $x = a$. Let $a_1, \ldots, a_n$ all converge to the limit a. Then since u always lies between the greatest and least of these $n + 1$ numbers, it also has the limit a. The right side of (4) then has the limit $f^n(a)/n!$. We *define* this to be the value of the integral (4) when $a_1, \ldots, a_n$ are equal to a, and write in symbols

$$\int_{(pq)} \frac{f(t)dt}{(t-a)^{n+1}} = \frac{f^n(a)}{n!}. \tag{7}$$

It is to be distinctly understood that this symbol means that $a_1 \ldots, a_n$ are to be put equal to a *after* the integration has been performed and not *before*.

The integral

$$\int_{(pq)} \frac{f(t)dt}{(t-a)^{n+1}}, \tag{8}$$

evaluated by the *ab initio* process of § 2 is *infinite* for all even integers n, and has no meaning. It is discarded from our consideration. For $n = 1$, however, we have by the *ab initio* process

$$\int_{(pq)} \frac{f(t)dt}{(t-a)^2} = \int_{(pq)} \left\{ \frac{f(t) - f(a)}{t-a} \frac{dt}{t-a} + f(a) \frac{dt}{(t-a)^2} \right\}$$

$$= f'(a),$$

as before, since by § 1, we have the indefinite integral

$$\int_{(pq)} \frac{dt}{(t-a)^{m+1}} = \frac{1}{-m} \left\{ \left[\frac{1}{\alpha^m} - \frac{1}{\beta^m} \right] + (-1)^m \left[\frac{1}{\alpha^m} - \frac{1}{\beta^m} \right] \right\}, \tag{9}$$

$$= 0, \text{ when } m \text{ is odd,}$$

$$= \frac{2}{m} \left[\frac{1}{\beta^m} - \frac{1}{\alpha^m} \right], \text{ for } m \text{ even.}$$

The latter is therefore infinite when $\alpha = 0$.*

In order that (8) should be the same as (7), it is necessary and sufficient that (9) should be 0 for all values of the integer m ‡ 0. This is not true for a real variable, as we have defined the integral, but is true for a complex variable under Cauchy's well known method.

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* It does not seem possible to make this integral zero by letting α and β converge to zero and their ratio to a finite real number different from 1.

With the definite meaning assigned to (7), however, we do have

$$\int_{(pq)} \frac{dt}{(t-a)^{n+1}} = 0$$

for all integers n, and equal to 1 for $n = 0$. Understanding clearly these conditions we can apply the process of circuit integration "across" any pole a of multiplicity n, that pole being regarded as the result of running together of n poles of order 1.

The applications of the method of circuit integration for a real variable are limited to the producing of certain finite-difference and differential-equations from the circuit integrals directly or from a given algebraic identity. Already we have had an illustration of this in the production of Lagrange's interpolation formula in (4) and of the result $\Delta^n f(x) = h^n f^n(u)$ of (6). This last lies at the foundation of the formation of differential equations from those in finite differences.

7. An important extension of (4) is as follows. If $\varphi(x) = (x - a_1)^{m_1+1} \ldots (x - a_n)^{m_n+1}$, and $f(x)$ is any one-valued and continuous function in $((p, q))$. Then

$$\int_{(pq)} \frac{f(t)dt}{\varphi(t)} = \frac{f(x)}{\varphi(x)} - \sum_{r=1}^{n} \frac{1}{m_r!} \left[\frac{d}{dt}\right]^{m_r} \left\{ \frac{f(t)}{\varphi(t)} \frac{(t-a_r)^{m_r+1}}{x-t} \right\}_{t=a_r} \tag{10}$$

Also

$$\left[\frac{d}{dt}\right]^{m_r} \left\{ \frac{f(t)}{\varphi(t)} \frac{(t-a_r)^{m_r+1}}{x-t} \right\}$$

$$= \sum_{p=0}^{m_r} C_{m_r, p} \frac{(-1)^p p!}{(x-a_r)^{p+1}} \left[\frac{d}{dt}\right]^{m_r-p} \left\{ (t-a_r)^{m_r+1} \frac{f(t)}{\varphi(t)} \right\}_{t=a_r}$$

If we multiply (10) by $\varphi(x)$ then the member on the right vanishes

$$N = m_1 + \ldots + m_n + n$$

times at $a_1, \ldots, a_n$, and since the rational integral function in x under the sigma sign is of a degree less than N, we have

$$\varphi(x) \int_{(pq)} \frac{f(t)}{\varphi(t)} dt = \frac{\varphi(x)}{N!} f^N(u)$$

when u lies between the greatest and least of $a_1, \ldots, a_n, x$. If $f(x)$ is a rational integral function of degree less than $\varphi(x)$, the right side of (10) is zero, and we have the partition of the rational function into partial fractions.

8. Making use of Euler's identity in the form

$$\frac{1}{t-x} = \sum_{r=0}^{n} \frac{\psi_r(x)}{\psi_r(t)} \frac{1}{x-a_r} + \frac{\psi_n(x)}{\psi_n(t)} \frac{1}{t-x}, \tag{11}$$

where $\psi_r(x) = (x - a_0) \ldots (x - a_r)$. Multiply this by $f(t)dt$ and take the circuit integral over an interval $((p, q))$ containing all the poles. The function $f(x)$ being as before one-valued and continuous in $((p, q))$, we have

$$f(x) = \sum_{r=0}^{n} \frac{\psi_r(x)}{\psi_r'(a_r)} \frac{f(a_r)}{x - a_r} + \psi_n(x) \int_{(pq)} \frac{f(t)\, dt}{(t - x)\, \psi_n(t)}. \tag{12}$$

The last term on the right is equal to

$$\frac{\psi_n(x)}{(n + 1)!} f^{n+1}(u),$$

u between the greatest and least of the a's and x.

If in this we put $a_0 = a$ and $a_p = a + ph$, and write a before $(x - a)^{(0)} = 1$ and

$$(x - a)^{(n+1)} = (x - a)(x - a - h) \ldots (x - a - nh),$$

(12) becomes

$$f(x) = \sum_{r=0}^{n} \frac{(x - a)^{(r)}}{r!} \frac{\Delta^r f(a)}{h^r} + \frac{(x - a)^{(n+1)}}{(n + 1)!} f^{n+1}(u). \tag{13}$$

If $h = 0$, (13) becomes

$$f(x) = \sum_{r=0}^{n} \frac{(x - a)^r}{r!} f^r(a) + \frac{(x - a)^{n+1}}{(n + 1)!} f^{n+1}(u), \tag{14}$$

provided $f(x)$ admits differentiation at a and in (ax).

9. This last result is also obtained by integrating the identity

$$\frac{1}{t - x} = \frac{1}{t - a} + \frac{x - a}{(t - a)^2} + \ldots + \frac{(x - a)^n}{(t - a)^{n+1}} + \frac{(x - a)^{n+1}}{(t - x)(t - a)^n}, \tag{15}$$

after multiplying by $f(t)dt$, according to the meaning assigned in (7).

Again,

$$\int_{(pq)} \frac{f(t)dt}{(t - x)(t - a)^{n+1}} = \frac{1}{n!} \left[\frac{d}{da}\right]^n \frac{f(a)}{a - x} + \frac{f(x)}{(x - a)^{n+1}}$$

which produces (14) independently. Also this same integral can be written equal to

$$\int_{(pq)} \frac{f(t) - f(x)}{(t - x)(t - a)^{n+1}}\, dt + f(x) \int \frac{dt}{(t - x)(t - a)^{n+1}}$$

which is equal to

$$\frac{1}{n!} \left[\frac{d}{da}\right]^n \left\{ \frac{f(x) - f(a)}{x - a} \right\},$$

giving another obvious identity.

10. If $\theta(x)$ θ_x be a function which does not vanish in $((p, q))$. Then

$$\int_{(pq)} \frac{dt}{(\theta_t - \theta_a)^{n+1}} = \int_{(pq)} \left[\frac{t-a}{\theta_t - \theta_a}\right]^{n+1} \frac{dt}{(t-a)^{n+1}}$$

$$= \frac{1}{n!}\left[\frac{d}{dt}\right]^n \left[\frac{t-a}{\theta_t - \theta_a}\right]^{n+1}_{t=a},$$

meaning $t = a$ after differentiation. Making use of the identity (15) in which we write $t = \theta_t$, $a = \theta_a$, then we can get Burmann's formula in a variety of forms. For example, multiply by $f'(t)dt$ and integrate (circuit) over (ax) and we have

$$\frac{f'(x)}{\theta'(x)} = \sum_{r=0}^{n} \frac{(\theta_t - \theta_a)^r}{r!} D_t^r \left\{ \left[\frac{t-a}{\theta_t - \theta_a}\right]^{n+1} f'(t) \right\}_{t=a}$$

$$+ \frac{(\theta_x - \theta_a)^{n+1}}{(n+1)!} D_t^{n+1} \left\{ \left[\frac{t-a}{\theta_t - \theta_a}\right]^{n+1} \frac{f'(t)}{\theta_t - \theta_x} \right\}_{t=u}$$

Multiply this by $\theta'(x)dx$ and integrate ordinarily from a to x and we get Burmann's with the remainder as a definite integral. Burmann's is produced directly with coefficients in a different form if we multiply (15) as above by $[f(t) - f(a)]\,\theta'(t)dt$ and circuit integrate over (ax).

Circuit Integration renders no assistance in the expansion of functions further than producing the finite forms of the Differential Calculus which do not lead to general, but merely to particular, solutions of that problem.